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THE IMAGINARY UNIT DESCRIBED AS AN ALGEBRAIC AMBIGUITY

ORIGINAL ARTICLE

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ABSTRACT

A complex number is formed by a real part and an imaginary part, the latter being composed of a real number multiplied by the imaginary unit "*i*," defined as the square root of -1. For this reason, imaginary numbers cannot be placed on the real number line. The geometric representation of complex numbers is done in the so-called complex plane, composed of a real axis and an imaginary axis, orthogonal to the real line. However, from a philosophical point of view, the fact that the imaginary axis is not identified with any axis in three-dimensional Euclidean space raises ontological questions that transcend the well-established mathematical formalism of the set of complex numbers: are imaginary numbers just a sophism, an arbitrary invention, or a true mathematical discovery? Are imaginary numbers a glimpse of a parallel reality that is beyond our perception? In this article, I seek to demonstrate that the imaginary unit can be interpreted as an algebraic ambiguity that arises when representing certain vectors in the real plane as one-dimensional variables. Consequently, we will see how it is possible to express complex numbers in terms of real numbers.

Keywords: Imaginary numbers, Complex numbers, Complex plane, Polynomial functions.

1. INTRODUCTION

Since their emergence in the second half of the 16th century to today, complex numbers have been important not only for mathematics but also have allowed scientists and engineers (Cayemitte, 2000) numerous applications that have helped drive scientific and technological development. Particularly in Physics, recent studies, Renou *et al*. (2021) and Li *et al*. (2022), have shown that complex numbers are indeed indispensable for expressing standard quantum theory. However, before the formal establishment of the complex number system, imaginary numbers were viewed with some suspicion by mathematicians and were initially treated only as a convenient algebraic trick to solve cubic equations where dealing with square roots of negative numbers was necessary. The origin of complex numbers is linked to the Italian mathematician Rafael Bombelli (1526-1572), whose work (Bombelli, 1572) demonstrated that manipulating $\sqrt{-1}$ using the common rules of arithmetic led to correct results. What followed from there was a long history that involved several generations of mathematicians, ranging from previously unknown names, such as Gaspar Wessel and Jean-Robert Argand, to renowned names such as René Descartes, John Wallis, Gauss, and Hamilton. For a detailed overview of this fascinating history, it is recommended to read the book "An imaginary tale: the story of $\sqrt{-1}$" (Nahin, 1998). However, far from a historical review, the aim here is to present a completely new interpretation for the imaginary unit that, if proven correct, could point to yet unexplored paths.

2. SCALAR RECTANGLES AND THE SCALAR PLANE

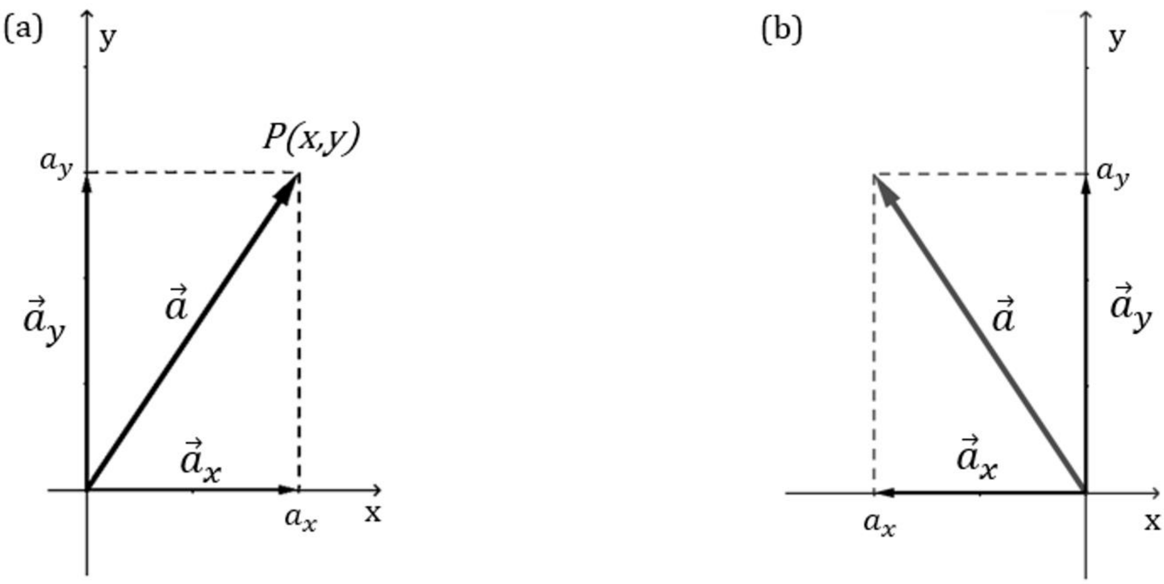
2.1 PRODUCT OF THE SCALAR COMPONENTS

Given a vector $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$, the quantities $a_x\hat{\imath}$ and $a_y\hat{\jmath}$ are called the vector components of vector $\vec{a}$, and the quantities a_x and a_y are called the scalar components, or simply components (Halliday; Resnick, 1991). By placing the origin of the vector at the origin of the coordinate system, we can use point $P = (x, y)$ at the end of the vector to define it, as seen in figure 1a. This allows us to write the vector in the form of an ordered pair $\vec{a} = (x, y)$. We will now define the product of the scalar components "A" of a given vector as:

$$A = (a_x)(a_y) = (x)(y). \tag{1}$$

We see that the vector $\vec{a}$ is diagonally inscribed in a rectangle whose sides, starting from the origin, are delimited by the scalars a_x and a_y. For this reason, we will call this rectangle a 'scalar rectangle.' As a direct consequence of definition (1), vectors in odd quadrants have positive products of components, while vectors in even quadrants have negative products, as seen in figure 1b. We can say that the scalar components a_x and a_y they are complementary to each other, in the sense that both determine the vector components that, when added, allow us to obtain the vector of interest and, consequently, the scalar rectangle associated with it.

Fig. 1: (a) Vector $\vec{a}$ with its components in the first quadrant; (b) Vector $\vec{a}$ and its components in the second quadrant: A negative



Source: Author (2024).

2.2 THE SCALAR PLANE

Consider the *xy* and *αγ* planes in three-dimensional Euclidean space. The *xy* plane will be used to plot the curve of any second-degree equation, while the *αγ* plane, hereinafter referred to as the "scalar plane," will be used to represent the terms of this equation through scalar rectangles and their associated vectors. For this, let us consider the quadratic polynomial function $y = f(x)$ and its development below:

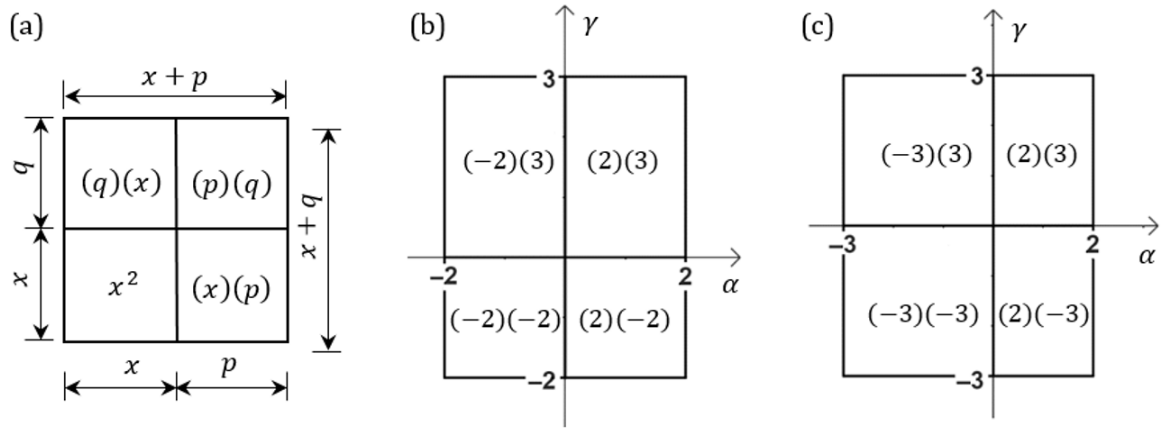
$$\text{y} = (x + p)(x + q)$$

$$y = (x)(x) + (q)(x) + (p)(x) + (p)(q), \tag{2}$$

$$y = x^2 + (p+q)x + (p)(q)$$

where *p* and *q* are real numbers. Making $y = 0$, we obtain a second-degree equation in which the values $x = x_1$ and $x = x_2$ are its roots. If we consider each term of the function seen in (2) as the product of the scalar components of a vector in the *αγ* plane, we will see that each of these products will be associated with a scalar rectangle, allowing us to obtain the generic diagram seen in figure 2a. Note in this figure that in the term x^2 the variable *x* forms the two perpendicular sides of the same square. Let's take the equation as an example $x^2 + 5x + 6 = 0$, whose roots are $x_1 = -2$ and $x_2 = -3$. It can be written as $(x)(x) + (3)(x) + (2)(x) + (2)(3) = 0$, which allows us to obtain figure 2b for the root x_1 and figure 2c for x_2.

Fig. 2: (a) Diagram for the quadratic function; (b) Representation of the function $y = x^2 + 5x + 6$ for $x = -2$; (c) The same representation for $x = -3$



Source: Author (2024).

3. AMBIGUITY AND THE JOINT NOTATION

3.1 THE SCALAR PLANE AND THE CONJUNCTION E

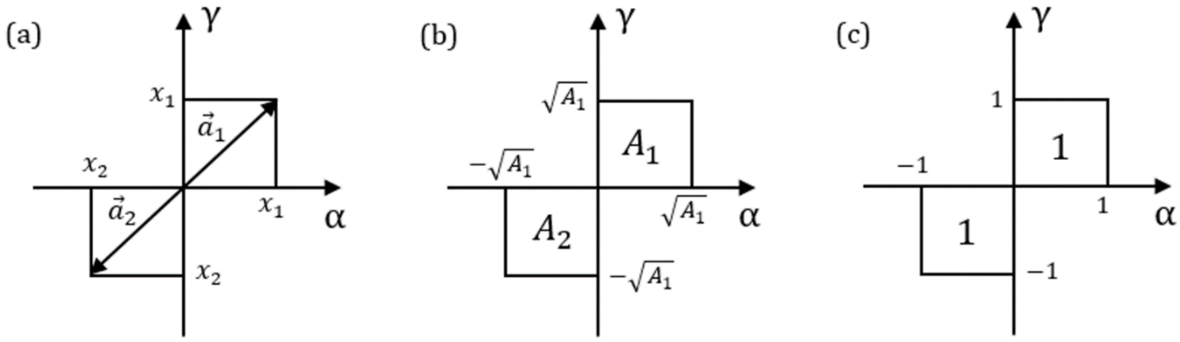
In the previous section, we saw that the variable *x* originates the two sides of a square scalar rectangle, referring to the quadratic term, when represented in the scalar plane. Therefore, in this plane, *x* occupies two dimensions, namely, one on the *α* axis and another on the *γ* axis. Thus, for any *x*, we have $\alpha = x$ on the horizontal axis and $\gamma = x$

on the vertical axis, as in the example of figure 3a, where we have a case in which the root x_1 is positive and x_2 is negative. Let's now proceed in the opposite direction: instead of obtaining a rectangle from the value of *x*, let's express this variable starting from the components of a vector $\vec{a} = (\alpha, \gamma)$ and the scalar rectangle associated with it. This can be done with the help of the conjunction E, as follows:

$$(x = \alpha) \wedge (x = \gamma). \tag{3}$$

Even though the variable *x* has numerically equal values on these two axes, there is an ambiguity as it simultaneously represents the two complementary components of the same vector.

Fig. 3: (a) Case where there is one root in each odd quadrant; (b) Each square has sides with length equal to the square root of the product of the components; (c) The roots of $x^2 - 1 = 0$. The value inside each rectangle is the product of the components of the associated vector



Source: author (2024).

Now let's see how the complementary values of *x* seen in the scalar plane correspond to the unique value that this variable presents in the *xy* plane. Note in figure 3b that each side of the square formed by the variable *x* has a length equal to $\sqrt{A}$, where A is the product of the components of the vector associated with that square. So, for $x = x_1$ in the first quadrant, the *α* and *γ* components are equal to $\sqrt{A_1}$, which gives us $(x = \sqrt{A_1}) \wedge (x = \sqrt{A_1})$, as seen in notation (3). In the third quadrant, we will have $\sqrt{A_2} = -\sqrt{A_1}$, which implies $(x = -\sqrt{A_1}) \wedge (x = -\sqrt{A_1})$. We can now apply the idempotent law of conjunction (Lipschutz, 1972) to obtain:

$$\left(x=\sqrt{A_1}\right)\wedge\left(x=\sqrt{A_1}\right)\equiv\left(x=\sqrt{A_1}\right),$$

$$\left(x=-\sqrt{A_1}\right)\wedge\left(x=-\sqrt{A_1}\right)\equiv\left(x=-\sqrt{A_1}\right).$$

As an example, let's do the same development for the roots $x_1 = 1$ and $x_2 = -1$ of the equation $x^2 - 1 = 0$, as illustrated in 3c, obtaining $(x = 1) \wedge (x = 1)$ in the first quadrant and $(x = -1) \wedge (x = -1)$ in the third quadrant. Applying the idempotent law, we have:

$$(x = 1) \wedge (x = 1) \equiv (x = 1), \tag{4}$$

$$(x = -1) \wedge (x = -1) \equiv (x = -1). \tag{5}$$

Therefore, the ambiguity that the variable *x* exhibits in the first and third quadrants of the scalar plane is reducible to the univocal value observed in the *xy* plane.

3.2 JOINT NOTATION

Before we proceed, let's simplify the notation seen in (3). From this point on, we will define the following notation:

$$(x = \alpha) \wedge (x = \gamma) \equiv_{def} x = (\alpha|\gamma). \tag{7}$$

We will call this form 'joint notation'. The expressions (4) and (5), respectively, can be written in the following joint forms:

$$x = (1|1) \equiv (x = 1), \qquad x = (-1|-1) \equiv (x = -1). \tag{8}$$

By setting *x* equal on both sides of each identity in (8), we have:

$$(1|1) = 1, \quad (-1|-1) = -1. \tag{9}$$

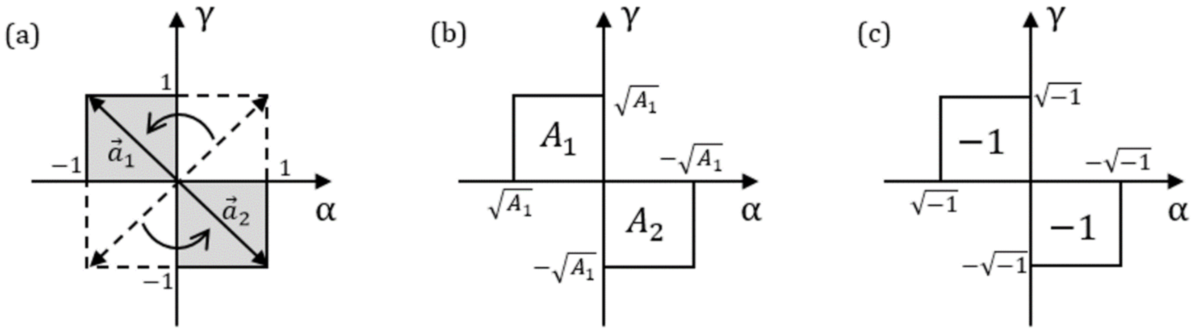
Let's consider the relative positions in which the scalar components appear on each side of the equivalence seen in (7) as a canonical order to be followed. Obviously, for any real *x*, there will always be a combination of components $(x|x)$ in the scalar plane. It is also worth noting that, with α and *γ* being the two coordinates of the same point in the plane, the notation $(\alpha|\gamma)$ is equivalent to the ordered pair notation (α,γ) and its use is justified here only to differentiate it from the notation (a,b) used to represent numbers in the complex plane.

4. IMAGINARY AND COMPLEX

4.1 THE IMAGINARY CONSTANT

According to what has been seen so far, the positive roots of second-degree equations allow us to form rectangles in the first quadrant of the scalar plane, while the negative roots do the same in the third quadrant. However, these diagrams still do not allow us to represent roots in the form of complex numbers or even purely imaginary numbers, since the *α* and *γ* axes are real. Hence, we could inquire about the second and fourth quadrants of this plane: do they have any relation to complex numbers? To answer this question, let's start from the roots of $x^2 - 1 = 0$, which have already been seen in figure 3c, and rotate the vectors associated with these roots around the origin, making a 90° angle, as shown in figure 4a. By doing this, we can see that each resulting vector now has one positive unitary component and one negative, both in the second and fourth quadrants. Let's assume that the same properties valid in the odd quadrants continue to hold in the even quadrants after the rotation, *i.e.*, the rectangle formed by the variable *x* continues to be a square with scalar product $A = (x)(x) = x^2$, where each side has a length equal to $\sqrt{A}$, see figure 4b.

Fig. 4: (a) Quadrant change by rotating the vectors by 90°; (b) The same relationships seen for the odd quadrants; (c) Scalar rectangles and their roots



Source: Author (2024).

So, if the root of the equation is in the second quadrant of the scalar plane, we will have *α* and *γ* equal to $\sqrt{A_1}$, which gives us $(x = \sqrt{A_1}) \wedge (x = \sqrt{A_1})$, as seen in notation (3). In the fourth quadrant, we will have $\sqrt{A_2} = -\sqrt{A_1}$, which implies $(x = -\sqrt{A_1}) \wedge (x = -\sqrt{A_1})$. However, as we now have $A_1 = -1$, each scalar component is $\sqrt{-1}$. See figure 4c. Therefore, we have $(x = \sqrt{-1}) \wedge (x = \sqrt{-1})$ in the second quadrant and $(x = -\sqrt{-1}) \wedge (x = -\sqrt{-1})$ in the fourth quadrant. Then, in the same way as we did for the odd quadrants, let's apply the idempotent law of conjunction:

$$\left((x = \sqrt{-1}) \wedge (x = \sqrt{-1})\right) \equiv (x = \sqrt{-1}), \tag{10}$$

$$\left((x = -\sqrt{-1}) \wedge (x = -\sqrt{-1})\right) \equiv (x = -\sqrt{-1}). \tag{11}$$

To apply (10) and (11) to the ambiguity condition of the quadratic term expressed by (3), we will substitute the real values of α and γ seen in figure 4a, obtaining:

$$\left((x = -1) \wedge (x = 1)\right) \equiv (x = \sqrt{-1}), \tag{12}$$

$$\left((x = 1) \wedge (x = -1)\right) \equiv (x = -\sqrt{-1}). \tag{13}$$

However, unlike what was done in section 3.1, it is no longer possible to apply the idempotent law of conjunction to assign a single value to the variable *x*; that is, in the even quadrants, the scalar components present an irreducible ambiguity. This means that there are combinations of components in the scalar plane for which there is no corresponding real number in the *xy* plane. However, we know that the solution $x = \pm\sqrt{-1}$ can also be written as $x = \pm i$, where *i* is the imaginary constant. Thus, replacing $\sqrt{-1}$ with *i* in (12) and (13), we get:

$$(x = -1) \wedge (x = 1) \equiv (x = i), \tag{14}$$

$$(x = 1) \wedge (x = -1) \equiv (x = -i). \tag{15}$$

Applying the joint notation to (14) and (15), we have:

$$\big(x = (-1|1)\big) \equiv (x = i), \tag{16}$$

$$\big(x = (1|-1)\big) \equiv (x = -i). \tag{17}$$

Finally, by eliminating *x* in (16) and (17) respectively, we get:

$$i = (-1|1), \qquad -i = (1|-1). \tag{18}$$

Therefore, note that the figure we obtained in 4a, by rotating the roots of $x^2 - 1 = 0$, by 90°, corresponds to the two roots of the equation $x^2 + 1 = 0$ in the scalar plane, namely $x_1 = i$ and $x_2 = -i$.

4.2 IMAGINARY NUMBERS

The rule for multiplying a real number by the imaginary unit can be understood as multiplying this real by a vector $\vec{a} = (-1, 1)$ in the scalar plane, that is, the multiplication will be component by component. Thus, if *k* is a real constant and $i = (-1|1)$, we have:

$$ki = (k)(i) = (k)(-1|1) = (-k|k)$$

However, in joint notation, $k = (k|k)$. Therefore, we can write:

$$ki = (k|k)(i) = (k|k)(-1|1) = (-k|k) \tag{19}$$

Letting k vary from $-\infty$ a $+\infty$, we obtain the sequence of imaginary numbers in the scalar plane. Here are some examples:

$$3i = (-3|3), \qquad -5i = (5|-5), \qquad \sqrt{2}\,i = \left(-\sqrt{2}\middle|\sqrt{2}\right), \qquad -\pi i = (\pi|-\pi).$$

4.3 COMPLEX NUMBERS

The operation of adding two real numbers, x_1 and x_2, consists of adding two vectors in the scalar plane, which is done by separately adding the components on each axis. Thus, if $x_1 = (\alpha_1|\gamma_1)$ and $x_2 = (\alpha_2|\gamma_2)$, he sum is given by:

$$x_1 + x_2 = (\alpha_1|\gamma_1) + (\alpha_2|\gamma_2) = (\alpha_1 + \alpha_2|\gamma_1 + \gamma_2). \tag{20}$$

For example: $2 + 3 = (2|2) + (3|3) = (5|5)$.

A complex number can be succinctly described as a real number added to a multiple of the imaginary unit. Thus, being *a* and *b* two real numbers and i the imaginary unit, a complex number can be algebraically represented as $a + bi$. Similar to a real number, an imaginary number represents a vector in the scalar plane, as seen in (19). Therefore, the operation of adding a real number to an imaginary number will be the same as when adding two real numbers. Thus, replacing x_1 in (20) with *a*, and x_2 with *bi*, we have:

$$a + bi = (a|a) + b(-1|1)$$

$$a + bi = (a|a) + (-b|b) \tag{21}$$

$$a + bi = (a - b|a + b) \tag{22}$$

Let's see some examples:

$2 + 5i = (2|2) + (-5|5) = (-3|7);$ $\qquad -4 + i = (-4|-4) + (-1|1) = (-5|-3);$

$3 - 2i = (3|3) + (2|-2) = (5|1);$ $\qquad -1 - i = (-1|-1) + (1|-1) = (0|-2).$

Therefore, in the scalar plane, a complex number represents the sum of a vector from an odd quadrant with a vector from an even quadrant. Also note that we have two ways to represent a complex number jointly: the form seen in (21) will be called the analytical joint form and the form seen in (22) will be called the synthetic joint form.

5. CHANGE FROM JOINT FORM TO ALGEBRAIC FORM

The change from a number written in algebraic form $x = a + bi$ to the synthetic joint form $x = (\alpha|\gamma)$ can be done by the equality seen in (22). For the reverse change, we just need to solve the system composed of $a - b = \alpha$ e $a + b = \gamma$, which results in:

$$(\alpha|\gamma) = \frac{\gamma + \alpha}{2} + \frac{\gamma - \alpha}{2}\, i \qquad (23)$$

For example, the complex number $(-17|7)$ can be written as $-5 + 12i$ in algebraic form:

$$(-17|7) = \frac{7 - 17}{2} + \frac{7 - (-17)}{2}\, i = \frac{-10}{2} + \frac{7 + 17}{2} i = -5 + 12i$$

6. MULTIPLICATION IN JOINT FORM

6.1 GENERAL FORMULA

Let's now derive a general formula for multiplying any two numbers written in synthetic joint form. First, given two complex numbers in algebraic form $z_1 = (a + bi)$ and $z_2 = (c + di)$, can find the product between them as follows:

$$z_1 * z_2 = (a + bi)(c + di) = ac + adi + bci + bdi^2$$

$$z_1 * z_2 = (ac - bd) + (ad + bc)i. \tag{24}$$

Using the transformation rule seen in (21), let's convert only the right side of the equality in (24) to joint form:

$$z_1 * z_2 = (ac - bd|ac - bd) + (-ad - bc|ad + bc)$$

$$z_1 * z_2 = (ac - bd - ad - bc|ac - bd + ad + bc) \tag{25}$$

In joint form, the product of two complex numbers can be written as:

$$z_1 * z_2 = (\alpha_1|\gamma_1)(\alpha_2|\gamma_2) \tag{26}$$

Replacing $z_1 * z_2$ with $(a + bi)(c + di)$ and converting the right side of (26) to algebraic form, as seen in (23), we get:

$$(a + bi)(c + di) = \left(\frac{\gamma_1 + \alpha_1}{2} + \frac{\gamma_1 - \alpha_1}{2}\ i\right)\left(\frac{\gamma_2 + \alpha_2}{2} + \frac{\gamma_2 - \alpha_2}{2}\ i\right) \tag{27}$$

Thus, by direct comparison, we obtain:

$$a = \frac{\gamma_1 + \alpha_1}{2}; \quad b = \frac{\gamma_1 - \alpha_1}{2}; \quad c = \frac{\gamma_2 + \alpha_2}{2}; \quad d = \frac{\gamma_2 - \alpha_2}{2}\,. \tag{28}$$

Substituting the values from (28) into (25), we get:

$$z_1 \cdot z_2 = \left(\begin{matrix} \left(\frac{\gamma_1+\alpha_1}{2}\right)\left(\frac{\gamma_2+\alpha_2}{2}\right) - \left(\frac{\gamma_1-\alpha_1}{2}\right)\left(\frac{\gamma_2-\alpha_2}{2}\right) \\ -\left(\frac{\gamma_1+\alpha_1}{2}\right)\left(\frac{\gamma_2-\alpha_2}{2}\right) - \left(\frac{\gamma_1-\alpha_1}{2}\right)\left(\frac{\gamma_2+\alpha_2}{2}\right) \end{matrix}\middle| \begin{matrix} \left(\frac{\gamma_1+\alpha_1}{2}\right)\left(\frac{\gamma_2-\alpha_2}{2}\right) - \left(\frac{\gamma_1-\alpha_1}{2}\right)\left(\frac{\gamma_2+\alpha_2}{2}\right) \\ +\left(\frac{\gamma_1+\alpha_1}{2}\right)\left(\frac{\gamma_2-\alpha_2}{2}\right) + \left(\frac{\gamma_1-\alpha_1}{2}\right)\left(\frac{\gamma_2+\alpha_2}{2}\right) \end{matrix}\right).$$

Finally, algebraically developing the two components of the equation above and substituting $z_1 * z_2$ for $(\alpha_1|\gamma_1)(\alpha_2|\gamma_2)$, we arrive at the general formula:

$$(\alpha_1|\gamma_1)(\alpha_2|\gamma_2) = \left(\frac{\alpha_2(\gamma_1+\alpha_1)-\gamma_2(\gamma_1-\alpha_1)}{2}\middle|\frac{\alpha_2(\gamma_1-\alpha_1)+\gamma_2(\gamma_1+\alpha_1)}{2}\right). \quad (29)$$

Examples:

$a)\ (-1|5)(-3|11) = \left(\frac{-3(5-1)-11(5+1)}{2}\middle|\frac{-3(5+1)+11(5-1)}{2}\right) = (-39|13).$

This is equivalente to $(2+3i)(4+7i) = -13+26i$.

$b)\ (-9|13)(13|-9) = \left(\frac{13(13-9)+9(13+9)}{2}\middle|\frac{13(13+9)-9(13-9)}{2}\right) = (125|125) = 125.$

This is equivalente to $(2+11i)(2-11i) = 125$.

6.2 MULTIPLICATION BY A REAL NUMBER

To obtain the product between a real number and a complex number in joint form, we simply do $\alpha_1 = \gamma_1 = k$, to get the joint form of the real number and $(\alpha_2|\gamma_2) = (\alpha|\gamma)$ for any complex number. Substituting into equation (29), we have:

$$(k|k)(\alpha|\gamma) = \left(\frac{\alpha(k+k)-\gamma(k-k)}{2}\middle|\frac{\alpha(k-k)+\gamma(k+k)}{2}\right)$$

$$(k|k)(\alpha|\gamma) = (k\alpha|k\gamma)$$

This result agrees with equation (19), which was used to generate any imaginary number from the imaginary unit.

6.3 PRODUCT OF TWO IMAGINARIES

A good simplification can be achieved when we have two imaginaries to be multiplied. Setting $\alpha_1 = -\gamma_1$ and $\alpha_2 = -\gamma_2$ and in formula (29), we get:

$$(\alpha_1|\gamma_1)(\alpha_2|\gamma_2) = \left(\frac{-\gamma_2(\gamma_1-\gamma_1)-\gamma_2(\gamma_1-(-\gamma_1))}{2}\middle|\frac{-\gamma_2(\gamma_1-(-\gamma_1))+\gamma_2(\gamma_1-\gamma_1)}{2}\right)$$

$$(\alpha_1|\gamma_1)(\alpha_2|\gamma_2) = \left(\frac{-\gamma_2(2\gamma_1)}{2}\middle|\frac{-\gamma_2(2\gamma_1)}{2}\right) = (-\gamma_2\gamma_1|-\gamma_2\gamma_1)$$

Optionally, we can eliminate the negative components of the result obtained by doing $-\gamma_1 = \alpha_1$ and $-\gamma_2 = \alpha_2$, yielding:

$$(\alpha_1|\gamma_1)(\alpha_2|\gamma_2) = (\gamma_1\alpha_2|\alpha_1\gamma_2);\;\; \alpha_1 = -\gamma_1,\;\; \alpha_2 = -\gamma_2\,. \qquad (30)$$

Examples:

$a)$ $(i)(i) = -1 \leftrightarrow (-1|1)(-1|1) = (-1|-1) = -1$;

$b)$ $(2i)(3i) = -6 \leftrightarrow (-2|2)(-3|3) = (-6|-6) = -6$;

$c)$ $(-4i)(4i) = 16 \leftrightarrow (4|-4)(-4|4) = (16|16) = 16$;

$d)$ $(3i)(-5i) = 15 \leftrightarrow (-3|3)(5|-5) = (15|15) = 15$;

e) $(-5i)(-5i) = -25 \leftrightarrow (5|-5)(5|-5) = (-25|-25) = -25$.

6.4 SQUARE OF A COMPLEX NUMBER

We can simplify formula (29) a bit by multiplying a complex number by itself. Setting $\alpha_1 = \alpha_2 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma$, yielding:

$$(\alpha|\gamma)^2 = (\alpha|\gamma)(\alpha|\gamma) = \left(\frac{\alpha(\gamma+\alpha)-\gamma(\gamma-\alpha)}{2}\middle|\frac{\alpha(\gamma-\alpha)+\gamma(\gamma+\alpha)}{2}\right)$$

$$(\alpha|\gamma)^2 = \left(\frac{\alpha^2+2\alpha\gamma-\gamma^2}{2}\middle|\frac{\gamma^2+2\alpha\gamma-\alpha^2}{2}\right) \qquad (31)$$

Examples:

$a)$ $(2+3i)^2 = -5+12i \leftrightarrow (-1|5)^2 = \left(\frac{(-1)^2+2(-1)(5)-25}{2}\middle|\frac{(5)^2+2(-1)(5)-(-1)^2}{2}\right) = (-17|7)$;

$b)$ $(3i)^2 = -9 \leftrightarrow (-3|3)^2 = \left(\frac{(-3)^3+2(-3)(3)-(3)^2}{2}\middle|\frac{(3)^3+2(-3)(3)-(-3)^2}{2}\right) = (-9|-9) = -9$;

$c)\ 2^2 = 4 \leftrightarrow (2|2)^2 = \left(\frac{2^2+2(2)(2)-2^2}{2}\middle|\frac{2^2+2(2)(2)-2^2}{2}\right) = (4|4) = 4.$

6.5 WHEN THE SQUARE IS EQUAL TO THE PRODUCT OF THE COMPONENTS

The square of a real number can be obtained in a simplified way by making $\alpha = \gamma = a$ equation (31), as follows:

$$(a|a)^2 = \left(\frac{a^2 + 2aa - a^2}{2}\middle|\frac{a^2 + 2aa - a^2}{2}\right) = (a^2|a^2) = (a)(a).$$

Similarly, we can obtain the square of an imaginary number in both forms $(-a|a)$ and the form $(a|-a)$ as follows:

$$(-a|a)^2 = \left(\frac{(-a)^2+2(-a)a-a^2}{2}\middle|\frac{a^2+2(-a)a-(-a)^2}{2}\right) = (-a^2|-a^2) = -a^2 = (a)(-a);$$

$$(a|-a)^2 = \left(\frac{a^2+2a(-a)-(-a)^2}{2}\middle|\frac{(-a)^2+2a(-a)-a^2}{2}\right) = (-a^2|-a^2) = -a^2 = (a)(-a).$$

Therefore, both the square of two real numbers and the square of two imaginary numbers can be obtained by the product of the scalar components.

6.6 WHEN THE SQUARE DIFFERS FROM THE PRODUCT OF THE COMPONENTS

Given $x = (\alpha|\gamma)$, with the components α and γ differing in magnitude, if we calculate x^2 by the product of the components as we did in 2.2, we will obtain a value different from that obtained by formula (31). For example, let $x = (-2|6)$, if we calculate x^2 as the product of the components, we obtain $x^2 = -12$. If we calculate x^2 as the square of a complex number, we find $x^2 = (-28|4)$. To understand why this occurs, consider $x = (\alpha|\gamma) = a + bi$. Raising *x* to the square in the analytic joint form, we have:

$$(\alpha|\gamma)^2 = ((a|a) + (-b|b))^2$$

$$(\alpha|\gamma)^2 = (a|a)^2 + 2(a|a)(-b|b) + (-b|b)^2$$

$$(\alpha|\gamma)^2 = (a^2|a^2) + 2(-ab|ab) + (-b^2|-b^2)$$

$$(\alpha|\gamma)^2 = (a^2 - b^2|a^2 - b^2) + 2(-ab|ab)$$

$$(\alpha|\gamma)^2 = \big((a+b)(a-b)\big|(a+b)(a-b)\big) + (-2ab|2ab)$$

Remembering that $\alpha = a + b$, $\gamma = a - b$ is that $a = \frac{\gamma+\alpha}{2}$ and $b = \frac{\gamma-\alpha}{2}$, we will have:

$$(\alpha|\gamma)^2 = (\alpha\gamma|\alpha\gamma) + \left(-2\left(\frac{\gamma+\alpha}{2}\right)\left(\frac{\gamma-\alpha}{2}\right)\middle|2\left(\frac{\gamma+\alpha}{2}\right)\left(\frac{\gamma-\alpha}{2}\right)\right)$$

$$(\alpha|\gamma)^2 = (\alpha\gamma|\alpha\gamma) + \left(-\left(\frac{\gamma^2-\alpha^2}{2}\right)\middle|\left(\frac{\gamma^2-\alpha^2}{2}\right)\right)$$

Finally, writing the result in algebraic form, we find that:

$$(\alpha|\gamma)^2 = \alpha\gamma + \left(\frac{\gamma^2-\alpha^2}{2}\right)i \qquad (32)$$

This result shows us that when we square a complex number, we obtain as a result the product of the scalar components plus an imaginary term, which can be described as half the difference of the squares of the components. Therefore, if $(\alpha|\gamma)$ is a real number $(\alpha = \gamma)$, or if it is a pure imaginary number $(\alpha = -\gamma)$, when we calculate its square there will be no imaginary part, and thus the square of the complex number calculated by formula (31) will be equal to the product of the components.

Example:

Given $x = (-2|6)$, its square obtained by (31) is $x^2 = (-28|4)$. This result can be obtained by (32):

$$(-2|6)^2 = (-2)(6) + \left(\frac{6^2-(-2)^2}{2}\right)i$$

$$(-2|6)^2 = -12 + 16i$$

$$(-2|6)^2 = (-12|-12) + (-16|16)$$

$(-2|6)^2 = (-28|4)$, as expected.

7. VERIFICATION OF AMBIGUITY IN SECOND DEGREE EQUATIONS

7.1 GEOMETRIC METHOD

Let's now verify the graphical solutions in the real plane for second-degree equations with negative roots. For this, let's take as an example the equation $x^2 - 4x + 125 = 0$, whose roots are $x_1 = 2 + 11i$ and $x_2 = 2 - 11i$. Now, writing x_1 in joint form, we obtain:

$$x_1 = 2 + 11i = (2|2) + (-11|11) = (2 - 11|2 + 11) = (-9|13).$$

The first step is to write this equation in the form of scalar products, similar to what was seen in section 2.2, and then separate the independent term, as follows:

$$x^2 - 2x - 2x + 125 = 0;$$
$$x^2 - 2x - 2x = -125$$

The independent term is nothing more than the product of the two roots, and its calculation was already seen in example *b* of section 6.1. The part on the left side of the equality allows us to obtain the diagram in figure 5a. In the diagram in figure 5b, we write x_1 in the form $(2 - 11|2 + 11)$ and inside each rectangle is the product of the scalar components. Note that the base of the rectangle assumes the value $2 - 11$ as it refers to the α-axis, while the value $2 + 11$ lies along its height as the component of the γ-axis. Using this figure as a reference and remembering that x^2 is given by the product of the components, let's see how the value of x_1 makes the equation true:

$$x^2 - 2x - 2x = -125$$

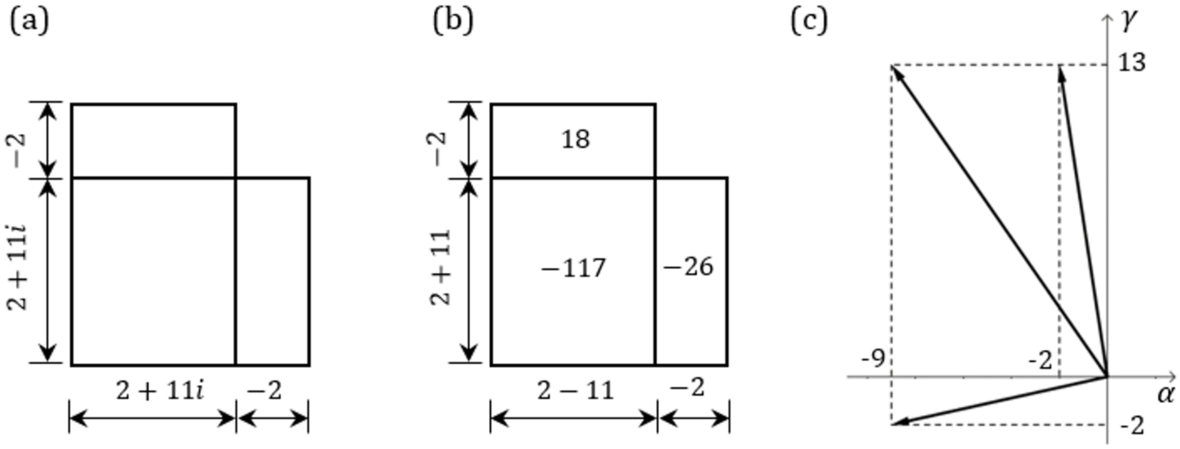
$$(2 - 11)(2 + 11) - 2(2 - 11) - 2(2 + 11) = -125$$

$$(-9)(13) - 2(-9) - 2(13) = -125$$

$$-117 + 18 - 26 = -125$$

$$-125 = -125$$

Fig.5: (a) Diagram for $x_1 = 2 + 11i$; (b) Diagram for $x_1 = (2|2) + (-11|11)$; (c) Vectors in the scalar plane for $x = x_1$



Source: Author (2024).

In figure 5c, we can see the combination of vectors in the scalar plane representing the equation $x^2 - 4x + 125 = 0$ when$x = x_1$. The same procedure can be applied to the root x_2, as follows:

$$x_2 = 2 - 11i = (2|2) + (11|-11) = (2 + 11|2 - 11) = (13|-9).$$

Substituting x_2 into the given equation, we have:

$$x^2 - 2x - 2x = -125$$

$$(2 + 11)(2 - 11) - 2(2 + 11) - 2(2 - 11) = -125$$

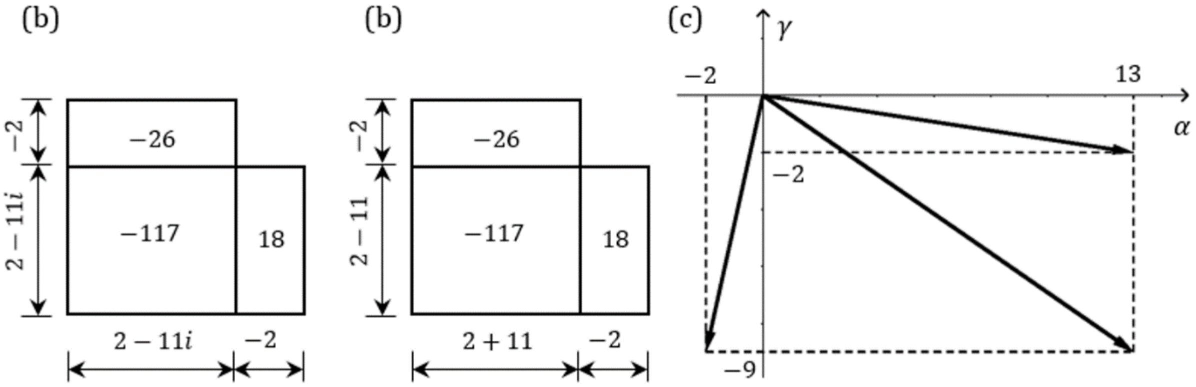
$$(13)(-9) - 2(13) - 2(-9) = -125$$

$$-117 - 26 + 18 = -125$$

$$-125 = -125$$

In figures 6a and 6b are the scalar diagrams made from x_2 and in figure 6c, is the combination of vectors that satisfy the given equation when $x = x_2$. Note, therefore, that the complex conjugates $2 + 11i$ and $2 - 11i$ are nothing more than the two rectangles obtained by permuting the α and γ components, both corresponding to the same product of components.

Fig.6: (a) Diagram for $x_2 = 2 - 11i$; (b) Diagram for $x_2 = (2|2) + (11|-11)$; (c) Vectors in the scalar plane for $x = x_2$



Source: author (2024).

7.2 ALGEBRAIC METHOD

Another way to verify the roots of a second-degree equation without using imaginary numbers is to simply replace the algebraic form with the joint form. Returning to the equation $x^2 - 4x + 125 = 0$, its roots in joint form are $x_1 = (-9|13)$ and $x_2 = (13|-9)$. Substituting the first root into the given equation, we have:

$$(-9|13)^2 - 4(-9|13) + 125 = 0$$

$$(-161|-73) + (36|-52) + (125|125) = 0$$

$$(-161 + 36 + 125|-73 - 52 + 125) = 0$$

$$0 = 0$$

Similarly, substituting the second root into the given equation, we have:

$$(13|-9)^2 - 4(13|-9) + 125 = 0$$

$$(-73|-161) + (-52|36) + (125|125) = 0$$

$$(-73 - 52 + 125|-161 + 36 + 125) = 0$$

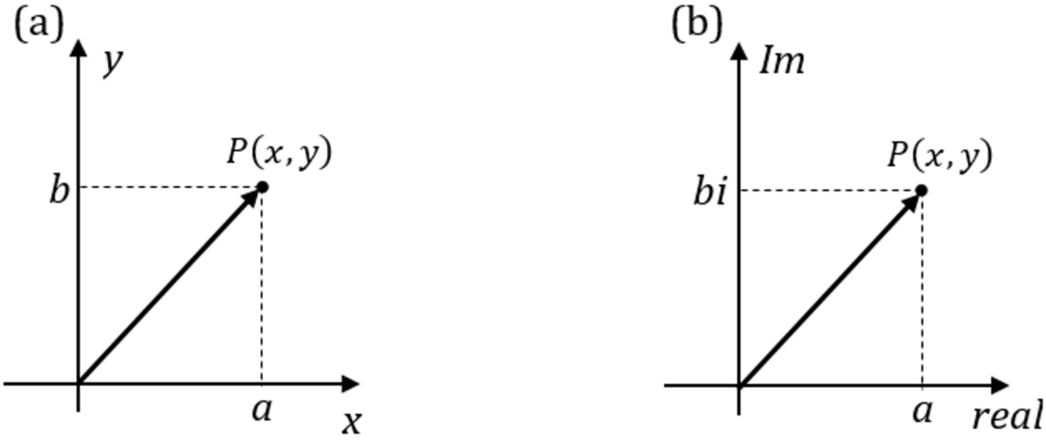
$$0 = 0$$

The basic difference between the two verification methods presented is that, in the first method, although the variable *x* takes on different values due to its ambiguous nature, each occurrence of *x* in the equation corresponds to a real and univocal value, following the diagram used as a reference. In the second method, the notation itself and the appropriate rules of algebraic manipulation already carry within them the ambiguous nature of the variable, so that every occurrence of *x* can always correspond to the same pair of scalar components. This second way of working with complex numbers, as we have seen, is completely equivalent to the algebraic form $x = a + bi$. Therefore, algebraically, writing $y = f(x)$, with $x \in \mathbb{C}$, instead of $y = f(\alpha, \gamma)$, with $(\alpha, \gamma) \in \mathbb{R}^2$**,** allows us to work with two-dimensional mathematical objects in a similar way to working with numbers in a single dimension.

8. COMPLEX PLANE VERSUS SCALAR PLANE

Complex numbers can be defined as ordered pairs (x, y) on which certain predefined operations are performed, so that for each point in the xy orthogonal plane, there is an associated complex number, see figure 7a. "Therefore, we can represent complex numbers on a coordinate plane in the same way we do when marking points on $\mathbb{R}^2$, observing, of course, the geometric meaning of such marking: that they are representing a number in the form $z = (a, b) = a + bi$ " (Molter *et al.*, 2020). Although in the notation $z = (x, y)$ both variables are real numbers, the variable y functions as a multiplier of the imaginary constant *i*, therefore, the complex plane has as its abscissa axis the real line and, as its ordinate axis, the imaginary line, as shown in figure 7b. The ordered pair notation only omits this information because it is inherent to the context of complex numbers in which it is used. Note, therefore, that there is a fundamental difference between the complex plane and the scalar plane: while the complex plane comprises a real axis and an imaginary axis, even if this is not explicitly shown in the notation, the scalar plane has exclusively real axes.

Fig.7: (a) Plane in $\mathbb{R}^2$ associated with the set of complexes; (b) Real axis and imaginary axis in the complex plane



Source: author (2024).

Therefore, the use of the scalar plane to represent complex numbers is not merely a formal matter, but a paradigm shift that allows us to understand complex numbers as part of the real plane. Thus, it is unnecessary to establish the set of complex numbers $\mathbb{C}$, since any element belonging to $\mathbb{C}$ is nothing more than an element of $\mathbb{R}^2$ and the imaginary unit i does not go beyond an interesting algebraic resource that allows us to operate in just one dimension with the elements belonging to $\mathbb{R}^2$.

9. CONCLUSION

The imaginary unit, although it can be operated on as a number, is not a number per se, but an ambiguous quantitative notion that simultaneously represents the two unit components of a vector located in the second quadrant of a real coordinate system, such that we cannot attribute a univocal value to i: all we can assert is that i^2 represents the product of the components of this vector, that is, $i^2 = -1$. Complex numbers, in turn, are algebraic representations of scalar rectangles with unequal sides. Each of these scalar rectangles results from the sum of two vectors, one located in an odd quadrant and the other in an even quadrant. This article constitutes only an alternative proposal for the interpretation of imaginary numbers and, consequently, complex numbers. Clearly, there are still points to be explored, such as the ambiguity in cubic equations and the trigonometric form of joint notation. However, these issues are beyond the scope of this article, and it would require much more space to treat them adequately. Due to the simplicity and effectiveness of the current treatment given to complex numbers, I believe that any effort made to develop the proposal presented here will have a primarily ontological motivation, although, from this new approach, results may arise that are useful not only for mathematics but also for all areas of knowledge that find applications for complex numbers.

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