METHODOLOGY OF THE ISOMETRIC POLAR CARTESIAN TRIGONOMETRIC CIRCLE IN THE GEOGEBRA SOFTWARE DEMONSTRATING THE RATIONALITY OF THE CONSTANT Π

ORIGINAL ARTICLE

SAMPAIO JUNIOR, Cloves Rocha


ABSTRACT

The article will present mathematical models demonstrating calculations of the relationships and proportions of the infinite periodic trigonometric circle using the GeoGebra software in the Cartesian, isometric, and polar planes. It will consider the first quadrant of the trigonometric cycle for the calculations of the trigonometric identities functions and apply the same reasoning calculations for the other quadrants. The infinite relationships and proportions between the circumferences' perimeters and their diameters, the angles (radian arcs) of the circumferences, and square roots. The oldest universal constant of mathematics, involved in circular symmetries, circular paths of stars and planets, in the propagation of electromagnetic fields, circles, and spheres and their relationships and proportions, are all approximated. Its relationship can be known exactly, or we must limit ourselves to the approximations of the calculations of the number Π. This procedure can be calculated with the help of computing, and trying to obtain its rational and periodic value with a ruler and compass will only lead to frustration. The demonstration carried out in the GeoGebra software, through mathematical models in the Cartesian, isometric, and polar planes, shows the interlinked calculations and theorems of the relationships and proportions with rational and periodic values of the infinite sums of the rational periodic calculations of the number Π (pi) (BECKMANN, 1971).

Keywords: Number Π (pi), Trigonometric circle, Radian angles, Polygons.
1. INTRODUCTION

Study of mathematics responsible for the relationship and proportion existing between the sides and angles of a right triangle. They have known fractional values represented for the relationships of sine, cosine, tangent, cotangent, cosecant, and secant. From the 15th century to the modernity of calculations and the creation of theoretical situations related to the study of angles in differential and integral calculus functions by scientists Isaac Newton and Leibniz, with definitive methods in the mathematical field, these are constantly employed in other sciences such as Medicine, Engineering, Physics, Chemistry, Geography, Astronomy, Biology, Cartography, and Navigation.

2. DEVELOPMENT

2.1 TRIGONOMETRIC CIRCUMFERENCE IN THE CARTESIAN PLANE

In the orthogonal Cartesian system, let's consider point A (0,1) on the A(x) axis, with an abscissa of 1. We then construct a circle with center at the origin O (0,0) of the system, passing through A, with a unit radius. We will conventionally consider point A as the origin of the oriented arcs of this circle, meaning that to traverse these arcs, point A will always be the starting point. Thus, given a plane $\alpha$, a point O (0,0), at a distance Radius ($r$), we have: $C: x^2 + y^2 = IR$ (real numbers).
Figure 1. Demonstration of the trigonometric circle in the Cartesian plane with radius=1

Source: Created by the author using GeoGebra (2021).
Figure 2. Demonstration of the trigonometric circle in the Cartesian plane with radius=0.5

Source: Created by the author using GeoGebra (2021).
Figure 3. Demonstration of the trigonometric circle in the Cartesian plane with radius=1.5

Source: Created by the author using GeoGebra (2021).
Figure 4. Demonstration of the trigonometric circle in the Cartesian plane with radius $= 2$

![Trigonometric Circle Diagram](image)

TRIGONOMETRIA DINÂMICA RACIONAL PERIÓDICA INFINITA

<table>
<thead>
<tr>
<th>TRIGONOMETRIA</th>
<th>RACIONAL</th>
<th>PERIÓDICA</th>
<th>Circunferências</th>
<th>Diâmetros</th>
<th>4</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ângulos</td>
<td>Seno</td>
<td>Cosseno</td>
<td>Tangente</td>
<td>Cotangente</td>
<td>Secante</td>
<td>Consecante</td>
</tr>
<tr>
<td>$0^\circ$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>$0.5$</td>
<td>$-0.866$</td>
<td>$1.7320508076$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$45^\circ$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$1.4142135624$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>$0.866$</td>
<td>$-0.5$</td>
<td>$1.7320508076$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
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<td>$0$</td>
<td>$\infty$</td>
</tr>
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<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
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<td>$-0.707$</td>
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<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$150^\circ$</td>
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<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>$1$</td>
<td>$-0$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$\infty$</td>
</tr>
<tr>
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<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
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<td>$-0.707$</td>
<td>$1.4142135624$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$240^\circ$</td>
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<td>$-0.5$</td>
<td>$1.7320508076$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$270^\circ$</td>
<td>$1$</td>
<td>$-0$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$300^\circ$</td>
<td>$0.866$</td>
<td>$-0.5$</td>
<td>$1.7320508076$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$315^\circ$</td>
<td>$0.707$</td>
<td>$-0.707$</td>
<td>$1.4142135624$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$330^\circ$</td>
<td>$0.866$</td>
<td>$-0.5$</td>
<td>$1.7320508076$</td>
<td>$2.8284271247$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$360^\circ$</td>
<td>$1$</td>
<td>$-0$</td>
<td>$0$</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).
Figure 5. Demonstration of the trigonometric circle in the Cartesian plane with radius=2.5
Figure 6. Demonstration of the trigonometric circle in the Cartesian plane with radius=3

Source: Created by the author using GeoGebra (2021).
Figure 7. Demonstration of the trigonometric circle in the Cartesian plane with radius=3.5

Source: Created by the author using GeoGebra (2021).
Figure 8. Demonstration of the trigonometric circle in the Cartesian plane with radius=4

Source: Created by the author using GeoGebra (2021).
Figure 9. Demonstration of the trigonometric circle in the Cartesian plane with radius $= 4.5$

Source: Created by the author using GeoGebra (2021).
Figure 10. Demonstration of the trigonometric circle in the Cartesian plane with radius=5

Source: Created by the author using GeoGebra (2021).
2.2 TRIGONOMETRIC CIRCUMFERENCE IN THE ISOMETRIC PLANE

The circumference is a set of points in a plane where the distance to a specific point in that plane is equal to a given distance. The given point is the center, and the given distance is the radius (r) of the circumference. Thus, given a plane \( \alpha \), a point O (0,0), at a distance Radius (r), we have: \( C: x^2 + y^2 = IR \) (real numbers).

Figure 11. Trigonometric circle in the isometric plane diameter= \( \sqrt{3} = 1.732\ 050... \)

Source: Created by the author using GeoGebra (2021).
Table 1. Demonstration in the isometric plane of infinite rational periodic fig. 11

<table>
<thead>
<tr>
<th>Circunferências</th>
<th>Diâmetros</th>
<th>Divisões racionais</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.441 398 092 702 66...</td>
<td>1.732 050 807 668...</td>
<td>π=3.141 592 653 689 389...</td>
</tr>
<tr>
<td>Raio</td>
<td>0.866 025 403 784.</td>
<td>IM(φ)= {y ∈ R</td>
</tr>
</tbody>
</table>

2.3 SPECIAL VALUES

From convenient right triangles, the definitions of sine, cosine, tangent, cotangent, cosecant, and (secant) allow for the following table of infinite notable periodic rational values (IEZZI, 1993).
2.4 TRIGONOMETRIC FUNCTIONS

The trigonometric functions on the trigonometric circle are associated with the four axes, where the trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant will be defined in the Cartesian plane.
Figure 12. The four axes of the complete trigonometric functions. C: $x^2 + y^2 = 1$

<table>
<thead>
<tr>
<th>Constante</th>
<th>C: $x^2 + y^2 = 1$</th>
<th>(3.1415926535893891754368732170691)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perímetros</td>
<td>{per. c:0,33} =</td>
<td>(6.28318507177877834308737464341382)</td>
</tr>
<tr>
<td>Diâmetros</td>
<td>{diâmetro c:0,33} =</td>
<td>(2.00000000000000000000000000000000)</td>
</tr>
<tr>
<td>Divisões (racionais)</td>
<td>{divisão. c:0,33} =</td>
<td>(3.1415926535893891754368732170691)</td>
</tr>
<tr>
<td>Seno (60°)</td>
<td>{Seno.} = $\sqrt{3}/2$</td>
<td>(0.86602540378438912467970760950490)</td>
</tr>
<tr>
<td>Cosseno (60°)</td>
<td>{cosseno.} =1/2</td>
<td>(0.50000000000000000000000000000000)</td>
</tr>
<tr>
<td>Tangente (60°)</td>
<td>{tangente} = $\sqrt{3}$</td>
<td>(1.73205080756978249359415921900981)</td>
</tr>
<tr>
<td>Cotangente (60°)</td>
<td>{cotg.} = $\sqrt{3}/3$</td>
<td>(0.57735026910385718407772459478667)</td>
</tr>
<tr>
<td>Cossecante (60°)</td>
<td>{cossec.} =2$\sqrt{3}/3$</td>
<td>(1.154700538378648026230715595800522)</td>
</tr>
<tr>
<td>Secante (60°)</td>
<td>{secante} =$\sqrt{4}$</td>
<td>(2.00000000000000000000000000000000)</td>
</tr>
<tr>
<td>Radianos (60°)</td>
<td>{Arcos} = $\pi/3$</td>
<td>(1.0471975511911345659491808047976)</td>
</tr>
<tr>
<td>Radianos (120°)</td>
<td>{Arc.} =2$\pi/3$</td>
<td>(2.09439510234293418938836160947952)</td>
</tr>
<tr>
<td>Radianos (240°)</td>
<td>{Arc.} =4$\pi/3$</td>
<td>(4.18870920478585222872491642894254)</td>
</tr>
<tr>
<td>Radianos (300°)</td>
<td>{Arc.} =5$\pi/3$</td>
<td>(5.23598775598231528590614553617818)</td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).
2.4.1 SINE

The definition of sine $\alpha$ as the ratio and proportion between the opposite cathetus to $\alpha$ and the hypotenuse of the right triangle. Therefore, one can define a function from $\mathbb{R}$ to $\mathbb{R}$, such that for each $x$, it associates $y = \sin \alpha$.

From the definition of the function $y = f(x) = \sin x$, it follows that the Domain: $D(F) = \mathbb{R}$; image: $\text{IM}(f) = \{y \in \mathbb{R} | -1 \leq y \leq 1\}$

Figure 13. Complete trigonometric functions sine $\alpha$ 60º C: $x^2 + y^2 = 1$

<table>
<thead>
<tr>
<th>Seno ($\alpha=60^\circ$)</th>
<th>Cateto oposto</th>
<th>$\sqrt{3} \over 2$ = {0.866025403784891246797079}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^2 =$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{1.0000000000000000000}</td>
<td>{0.86602540378489124}</td>
<td>{0.5000000000000000}</td>
</tr>
<tr>
<td>$B^2 =$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{1.0000000000000000000}</td>
<td>{0.7500000000000000}</td>
<td>{0.2500000000000000}</td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).
2.4.2 COSINE

The definition of cosine $\alpha$ as the ratio and proportion between the adjacent cathetus to $\alpha$ and the hypotenuse of the right triangle. Therefore, one can define a function from $\mathbb{R}$ to $\mathbb{R}$, such that for each $x$, it associates $y=\cos \alpha$.

From the definition of the function $y=f(x)=\cos x$, it follows that the Domain: $D(F)=\mathbb{R}$; image: $\text{IM} (f\ x) = \{x \in \mathbb{R} \uparrow \ -1 \leq x \leq 1\}$

Figure 14. Demonstration of the four axes of the complete trigonometric functions. Cosine ($\alpha=60^\circ$) C: $x^2 + y^2 = 1$
2.4.3 TANGENT

The definition of tangent $\alpha$ as the ratio and proportion between the opposite cathetus ($\alpha$) and the adjacent cathetus ($\alpha$), and the hypotenuse of the right triangle.

Figure 15. Demonstration of the four axes of the complete trigonometric functions. Tangent ($\alpha=60^\circ$) C:

\[ x^2 + y^2 = 1 \]
2.4.4 COTANGENT

The definition of cotangent (α) is the inverse of the tangent, being the ratio and proportion between the adjacent cathetus (α) and the opposite cathetus (α), and the hypotenuse of the right triangle.

Figure 16. Demonstration of the four axes of the complete trigonometric functions. Cotangent (α=60º)

\[ C: x^2 + y^2 = 1 \]

Source: Created by the author using GeoGebra (2021).
2.4.5 COSECANT

The definition of cosecant (α) is the inverse of sine, being the ratio and proportion between (α), the hypotenuse, and the opposite cathetus (α).

Figure 17. Demonstration of the four axes of the trigonometric functions. Cosecant: $C: x^2 + y^2 = 1$

<table>
<thead>
<tr>
<th>Cotangente (α=60°)</th>
<th>Cateto adjacent</th>
<th>Cateto oposto</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$=0.5773502691907518487...$</td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).

2.4.6 SECANT

The definition of secant (α), as the reciprocal of cosine, corresponds to the ratio and proportion of the hypotenuse by the adjacent cathetus.

<table>
<thead>
<tr>
<th>Cossecante (α=60°)</th>
<th>hipotenusa</th>
<th>Cateto oposto</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2\sqrt{3}/3$</td>
<td>$1.154700538378648...$</td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).
Figure 18. Demonstration of the four axes of the trigonometric functions. Secant: $x^2 + y^2 = 1$

<table>
<thead>
<tr>
<th>Secante (α=60º)</th>
<th>hipotenusa</th>
<th>Cateto adjacente</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sqrt{4} = 2.00000000000000000000000000000000$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hipotenusa</th>
<th>Seno a (60º)</th>
<th>Cosseno (60º)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^2 =$</td>
<td>$B^2 =$</td>
<td>$C^2 $</td>
</tr>
<tr>
<td>${2.3094010767585}$</td>
<td>${1.154700538378648}$</td>
<td>${2.2}$</td>
</tr>
<tr>
<td>${5.33333333333}$</td>
<td>${1.33333333333}$</td>
<td>${4.}$</td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).

2.5 IRRATIONAL NUMBER $\pi$

Its representation, by the Greek letter $\pi$ (pi), comes from the Greek word for perimeter ($\pi\tau\rho\varepsilon\iota\miu\varepsilon\tau\rho\omicron\varsigma$), and was introduced in 1706 by William Jones (1676-1749), and popularized by Leonhard Euler (1707-1783). Mathematicians from various eras have tried to find a rationality for the constant $\pi$. The proof that the constant $\pi$ is an irrational...
number was made by Johann Lambert in 1761 and Legendre in 1794. The number $\pi$ is a transcendental number, which was proved by Ferdinand Lindemann in 1882.

### 2.5.1 CIRCUMFERENCE PERIMETER

The circumference's length, that is, its perimeter $c$, can be calculated using the equation. $C = \pi \cdot d = 2 \cdot \pi \cdot r$.

$\pi$ (pi) constant = 3.141 592 653 589 389 171 543 687 321 706 908 213...

### 2.5.2 RATIONALITY OF THE CONSTANT $\Pi$

The circle is the internal area ($A$), delimited on the circumference, which can be calculated using the equation. $\text{Área}(a) = \pi \cdot r^2$.

Area ($2 \cdot \pi$) = 6.283 185 307 178 778 343 087 374 464 314 816 427 962...

Figure 19. Demonstration and comparisons of the trisection of the circumferences' perimeters
The search for the rationality of \( \pi \), some historical approaches

<table>
<thead>
<tr>
<th>Year</th>
<th>Source</th>
<th>Country</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900 a. C.</td>
<td>Papiro Ahmes</td>
<td>Egípcio</td>
<td>( \frac{28}{34} \times 3,1605 )</td>
</tr>
<tr>
<td>1600 a. C.</td>
<td>Tablet Susa</td>
<td>Babilônio</td>
<td>( \frac{25}{8} \times 3,125 )</td>
</tr>
<tr>
<td>600 a. C.</td>
<td>Na Bíblia (Reis I, 7:23)</td>
<td>Feijão</td>
<td>3.00</td>
</tr>
<tr>
<td>500 a. C.</td>
<td>Bandhayana</td>
<td>Índia</td>
<td>3.09</td>
</tr>
<tr>
<td>250 a. C.</td>
<td>Arquimedes de Siracusa</td>
<td>Grego</td>
<td>3 ( \frac{10}{71} ) e 3 ( \frac{1}{7} )</td>
</tr>
<tr>
<td>150</td>
<td>Cláudio Ptolomeu</td>
<td>Greco egípcio</td>
<td>377/120-3.14166...</td>
</tr>
<tr>
<td>263</td>
<td>Liu Hui</td>
<td>China</td>
<td>3.14159</td>
</tr>
<tr>
<td>263</td>
<td>Ventilador Wang</td>
<td>China</td>
<td>157/50 x 3,14</td>
</tr>
<tr>
<td>300</td>
<td>Chang Hong</td>
<td>China</td>
<td>101/2 x 3,1623</td>
</tr>
<tr>
<td>500</td>
<td>Para Chongzhi</td>
<td>China</td>
<td>3.1415926 - 3.14159</td>
</tr>
<tr>
<td>500</td>
<td>Aryabhata</td>
<td>Índia</td>
<td>3.1416</td>
</tr>
<tr>
<td>600</td>
<td>Brahma Gupta</td>
<td>Índia</td>
<td>101/2 x 3,1623</td>
</tr>
<tr>
<td>800</td>
<td>Al-Khurisimi</td>
<td>Perdido</td>
<td>3.1416</td>
</tr>
<tr>
<td>1220</td>
<td>Fibonacci</td>
<td>Italiano</td>
<td>3.141818</td>
</tr>
<tr>
<td>1400</td>
<td>Madhava</td>
<td>Índia</td>
<td>3.14159265359</td>
</tr>
<tr>
<td>1424</td>
<td>Al-Kashi</td>
<td>Perdido</td>
<td>2( \pi = 6.2831853071 )</td>
</tr>
</tbody>
</table>

Fonte: Número \( \pi \) – Wikipédia, enciclopédia livre, (2022).
2.5.3 CLASSICAL METHODS OF BABYLONIA (2000 B.C.)

Archaeologists have been working systematically in Mesopotamia since before the mid-19th century and have unearthed over half a million clay tablets, of which four hundred have been identified as strictly mathematical. Museums in Paris, Berlin, and London, as well as some universities like Yale, Columbia, and Pennsylvania, have excellent collections of these tablets, which vary in size and are approximately one and a half centimeters thick. They are often rounded in shape, with cuneiform inscriptions on only one of their faces, and sometimes on both.

The number of continuous fractions $\pi$ inhabits approximately

$$\frac{6 \times L}{2 \times \pi \times L} \approx 0.96 \leftrightarrow \frac{3}{\pi} \approx 0.96 \leftrightarrow (\pi \approx \frac{25}{8} \approx 3.125...$$

The number of the continuous fraction $\pi$, infinite periodic rational

$$\frac{6 \times L}{2 \times \pi \times L} = 0.95492... \leftrightarrow \frac{3}{\pi} = 0.95492... \leftrightarrow (\pi = \frac{25.1327412287...}{8.0000000000...} = \pi)$$
Figure 20. Demonstration of the division of the classical Babylonian method of the circumference

| Círculos   | {Perímetro c.30}   | = 25.1327412287185230659355502185 |
| Diâmetro   | {Diâmetro. 30}    | = 8.0000000000000000000000000000 |
| Divisão    | {Constante p,30}  | = 3.14159265358938917514368732171 |

Métodos de cálculos por polígonos com valores aproximados irracionais

| Polígonos     | {lados,30}     | ≈ 4.00000000000000000000000000000 |
| Círculos      | {Perímetro c,30} | ≈ 25.00000000000000000000000000000 |
| Perímetros    | {Lados* p,06}  | ≈ 24.00000000000000000000000000000 |
| divisão       | {Perímetro/ Lados*06} | ≈ 1.00000000000000000000000000000 |

Métodos de cálculos por arcos de circunferência com valores exatos racionais

| Arcos Ângulos | {Radianos,30} | = 4.18879020478585222872491642895 |
| Círculos      | {Perímetro c, 30} | =25.1327412287185230659355502185 |
| Perímetros    | {Rad. 30 *06}  | = 25.1327412287185230659355502185 |
| Divisão racional | {Perímetro/ Lados*06} | =0.00000000000000000000000000000 |

Source: Created by the author using GeoGebra (2021).

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2.5.4 CLASSICAL METHODS OF THE RHIND PAPYRUS BY AHMES (1650 B.C.)

The Egyptian scribe Ahmes, 2000 B.C. Most of these problems come from measurement formulas necessary for calculating land areas and barn volumes. One of these records states that the area of a circle is equal to a square whose side is the diameter. \( \left( \frac{8}{9} \right) \) The continued fraction of \( \pi \) approximately inhabits:

\[
\pi \times r^2 = (3,160493 \times \frac{8}{9} 2r) \rightarrow \pi \approx \frac{256}{81} \approx 3.160...
\]

Figure 21. The classical methods of the Rhind papyrus with rational values
2.5.5 ARCHIMEDES OF SYRACUSE (250 B.C.)

The Greek mathematician Archimedes of Syracuse (287-212 B.C.) discovered a highly efficient method for obtaining sequences of approximations of the constant \( \pi \). In his work On the Measurement of the Circle, he developed a method of successive approximations for calculating the circumference of a circle.

He constructed regular polygons inscribed and circumscribed and divided the perimeter of each by the diameter of the circle, his studies with regular hexagons and doubling the number of sides of these polygons until reaching a polygon of 96 sides.
Seja $x$ um número racional. Assim, consideremos a fração contínua $x$, convergente e ilimitada, que a e b são números reais para $b \neq 0$.

### Circunferências $=223.$

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

### Diâmetros $=71.$

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>223</td>
<td>131</td>
</tr>
<tr>
<td>71</td>
<td>025</td>
</tr>
</tbody>
</table>

### Divisões

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi = 3.141$</td>
<td></td>
</tr>
</tbody>
</table>

Source: Created by the author using GeoGebra (2021).
Seja $\pi$ um número racional. Assim, consideremos a fração $\pi$, convergente e ilimitada, de que a e b são números reais para $b \neq 0$.

<table>
<thead>
<tr>
<th>Circunferências = 22.</th>
<th>Diâmetros=7.</th>
<th>Divisões</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 + \frac{1}{7}$</td>
<td>$\frac{22.02...}{7.01...}$</td>
<td>$\pi=3,141...$</td>
</tr>
</tbody>
</table>
### 2.6 ARCS AND ANGLES OF A CIRCUMFERENCE (DEGREE AND RADIANS)

The circumference in which two points A and B are taken. The circumference will be divided into two parts called arcs. Points A and B are the endpoints of these arcs. When A and B coincide, one of these arcs is called null and the other, an arc of one turn; we will say that the null arc measures 0° and the arc of one turn measures 360°. This way:

\[
1 \text{ grau}(^\circ) = \frac{1}{360} \quad \text{of the arc of one turn. As submultiples of the degree, we have: 1 minute (1') = } \frac{1}{60} \text{ of the degree, or 60 minutes = 1 degree (60' = 1°), and 1 second (1") = } \frac{1}{60} \text{ of the minute, or 60 seconds = 1 minute (60" = 1').}
\]

A radian is the measure of an arc that corresponds to the same length as the radius (r) of the circumference in relation to the central angle.

An arc of one radian (1 rad) is one whose length is equal to the radius of the circumference. \( C = 2\pi r = 360° = 2\pi \) radianos (NETO, 1978).
Figure 24. Demonstration of the dynamic trigonometric circle infinite rations

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Figure 25. Demonstration between circles and their diameters rational values

Source: Created by the author using GeoGebra (2021).
Figure 26. Demonstration between circles and their diameters rational values

Source: Created by the author using GeoGebra (2021).
Figure 27. Demonstration between circles and their diameters rational values

Source: Created by the author using GeoGebra (2021).
Figure 28. Demonstration between circles and their diameters rational values

Source: Created by the author using GeoGebra (2021).
Figure 29. Demonstration between circles and their diameters rational values

Source: Created by the author using GeoGebra (2021).
Figure 30. Demonstration between circles and their diameters rational values
Figure 31. Demonstration between circles and their diameters rational values

Source: Created by the author using GeoGebra (2021).
Figura 32. Demonstração entre circunferências e seus diâmetros valores racionais

Source: Created by the author using GeoGebra (2021).
Figure 33. Demonstration of rational division proving with calculations (0.9999...≠1)

Source: Created by the author using GeoGebra (2021).
2.7 PROOFS OF THE PYTHAGOREAN THEOREM AND THE RATIONAL SQUARE ROOT $\sqrt{2}$

The Pythagorean theorem is named after the Greek mathematician Pythagoras (570 B.C.-495 B.C.). The theorem's definition is a mathematical relationship between the lengths of the sides of a right triangle. The area of the square whose side is the hypotenuse is the sum of the areas of the squares whose sides are the catheti $a^2 + b^2 = c^2$.

2.8 THE DIAGONAL (D) OF THE SQUARE

Pythagoras knew at the time, however, that his theorem had a flaw. When the legs of the triangle were equal, his theorem did not work because there would be no irrational measure for the hypotenuse $\sqrt{2}$. Mathematicians who succeeded him since that time tried to understand why the sides of the triangle did not have a common measure, they could not be measured exactly through a unit common to both.

Indeed, suppose the hypotenuse is (p), and the legs are equal and represented by the letter (q). It is known that $p / q$ is an irreducible fraction, leaving an even numerator and an odd denominator, or vice versa. The application of the theorem results in $p^2 = 2q^2$.

Obviously, $(p^2)$ is even, as it is twice $(q^2)$, that is, it comes from a product of a number multiplied by 2. If $p^2$ is even, it implies that $p$ is even (the square of an even number is always even, and the square of an odd number is always odd), so $q$ must be odd, otherwise the given fraction would not be irreducible. Now, let's make $p = 2k$, since $p$ is even, we rewrite: $(2k)^2 = (2q^2)$, simplifying results in $(4k^2 = 2q^2)$ which results in $(2k^2 = q^2)$.

Let's calculate the diagonal (d) of the square as a function of the side L. The problem can also be formulated as follows: given the side L, calculate the diagonal (d). Applying the Pythagorean theorem, we can calculate the hypotenuse from the squares of the legs $d^2 = 1^2 + 1^2 \iff d = \sqrt{2}$.
Figure 34. Demonstrations of rational results of the Pythagorean theorem
2.9 THE DIAGONAL (D) OF THE CUBE

Let's calculate the diagonal (D) of the cube as a function of the side L. Applying the Pythagorean theorem to the triangle, we have $d=a\sqrt{3}=1.732...$
Figure 35. Demonstrations of the results of the Pythagorean theorem: \( (p^2 = 2q^2) \)

\[
\begin{align*}
\frac{p}{q} &= 2 \\
\frac{1.73205080756978249359415921901}{0.866025403784891246797079609505} &= 2. \\
a^2 &= b^2 + c^2 \\
(1.7320508072)^2 &= (1.414213562372)^2 \\
&= 1 \\
\text{resultados} &= 3 \\
&= 2 \\
&= 1
\end{align*}
\]

\[
\begin{align*}
\frac{p}{q} &= 2 \\
\frac{3.46410161513788456793638140788}{1.73205080756978249359415921901} &= 2. \\
a^2 &= b^2 + c^2 \\
(23.464101615)^2 &= (2.82842712474)^2 \\
&= 2 \\
\text{resultados} &= 12 \\
&= 8 \\
&= 4
\end{align*}
\]

\[
\begin{align*}
\frac{p}{q} &= 2 \\
\frac{5.19615242270682685190457211183}{2.5980762113525664597219395213} &= 2. \\
a^2 &= b^2 + c^2 \\
(25.1961524227)^2 &= (4.24264068711)^2 \\
&= 2 \\
\text{resultados} &= 27 \\
&= 18 \\
&= 9
\end{align*}
\]

\[
\begin{align*}
\frac{p}{q} &= 2 \\
\frac{6.92820323027576913587276281577}{3.46410161513788456793638140788} &= 2. \\
a^2 &= b^2 + c^2 \\
(26.9282032302)^2 &= (5.6568542499491)^2 \\
&= 2 \\
\text{resultados} &= 48 \\
&= 32 \\
&= 16
\end{align*}
\]

\[
\begin{align*}
\frac{p}{q} &= 2 \\
\frac{8.66025403784471141984095351972}{4.33012701892235570992047675986} &= 2. \\
a^2 &= b^2 + c^2 \\
(28.6602540378)^2 &= (7.07106781186)^2 \\
&= 2 \\
\text{resultados} &= 75 \\
&= 50 \\
&= 25
\end{align*}
\]

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\[
\begin{array}{c|c|c}
\hline
p & q & a^2 = b^2 + c^2 \\
\hline
10.3923048454136537038091442237 & 5.19615242270682685190457211183 & (210.392304845) \\
\hline
12.1243556529825959877773349276 & 6.06217782649129799388866746380 & (212.124355652) \\
\hline
13.8564064605515382717455256315 & 6.92820323027576913587276281577 & (213.85640646) \\
\hline
15.5884572681204805557137163355 & 7.79422863406024027785685816774 & (215.58845726) \\
\hline
17.3205080756894228396819070394 & 8.66025403784471141984095351972 & (217.320508075) \\
\hline
\end{array}
\]

Source: Created by the author using GeoGebra (2021).
Figure 36. Trigonometric circle at (3 d) of atomic rational values

Source: Created by the author using GeoGebra (2021).

Figure 37. Demonstration of trigonometric circle in (3 d) of rational values

Source: Created by the author using GeoGebra (2021).
3. FINAL CONSIDERATIONS

Mathematicians from various times sought to find a rationality for the constant π. However, they came to an incredible discovery for the time of the existence of irrational numbers. The proof that the constant π is irrational was made by Johann Lambert in 1761 and Legendre in 1794. In addition to being irrational, π is a transcendental number, which was proved by Ferdinand Lindemann in 1882. This means that there is no polynomial with integer or rational coefficients. The decimal representation is unpredictable.

The results of the demonstrations of the Cartesian, isometric, and polar mathematical models, with infinite periodic rational calculations, can be applied in all sciences, geometries of circular and spherical bodies, physics, and astronomy.

REFERENCES


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1 High school completed. ORCID: 0009-0001-1467-1150.